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M-ISOMETRIC WEIGHTED SHIFTS

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Abstract. In this paper, we characterize the m-isometric weighted shifts, using this characterization, we study the relations between the hyponormality and the m-isometricity of operators.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. An operator $T \in \mathcal{L}(\mathcal{H})$ is called an *m-isometry* if

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*k}T^k = 0. \tag{1.1}$$

It is easy to see that the equation (1.1) is equivalent to the following equation

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}||T^k x|| = 0 \tag{1.2}$$

for all $x \in \mathcal{H}$. If $m = 1$ then it is said to be *isometry.* It is easy to see that any m-isometric operator is also (m+1)-isometry. But the converse is not true in general. m-isometric operators was first introduced in ([1],[2],[3]) and has received much attention in recent years. Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*),

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the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

In this paper, we characterize the m-isometric weighted shifts, using this characterization, we study the relations between the hyponormality and the m-isometricity of operators.

2. Main results

We start from a basic result.

PROPOSITION 2.1. *Every isometry is subnormal and hence hyponormal.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$ is isometry then $T^*T = I$. By a direct calculation, we can see that $N := \begin{pmatrix} T & I - TT^* \\ 0 & T^* \end{pmatrix}$ is a normal extension of T. Thus, T is subnormal. $\square$

In view of Proposition 2.1, it is interesting to ask that every m-isometric operator is whether subnormal or hyponormal. To answer for this questions, we first give a characterization of m-isometric weighted shifts.

THEOREM 2.2. *W_α is m-isometry if and only if*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \gamma_{n+k} = 0 \tag{2.1}$$

for all $n \geq 0$.

Proof. ($\Rightarrow$) Suppose that W_α is m-isometry. Then we have the equation (2.1) from the equation (1.2) taking $x = e_n$.

($\Leftarrow$) Suppose that the equation (2.1) holds for all $n \geq 0$. Since $W_\alpha^* W_\alpha$ is diagonal, it is easy to show that the equation (1.2) holds for any $x = \sum_{n=1}^\infty x_n e_n \in \mathcal{H}$. Therefore, W_α is m-isometry. $\square$

COROLLARY 2.3. *For a weighted shift W_α, we have:*

(i) *W_α is isometry if and only if $\alpha_n = 1$ for all $n \geq 0$, i.e., W_α is the unilateral shift.*

(ii) W_α *is 2-isometry if and only if* $\alpha_{n+1}^2 = 2 - \frac{1}{\alpha_n^2}$ *for all* $n \geq 0$.

(iii) W_α *is 3-isometry if and only if* $\alpha_{n+2}^2 = 3 - \frac{3}{\alpha_{n+1}^2} + \frac{1}{\alpha_n^2\alpha_{n+1}^2}$ *for all* $n \geq 0$.

Proof. (i) If W_α is isometry then $\gamma_n = \gamma_{n+1}$ for all $n \geq 0$, and hence $\alpha_n = 1$ for all $n \geq 0$. The converse is clear.
(ii) Note that W_α is 2-isometry if and only if $\gamma_n - 2\gamma_{n+1} + \gamma_{n+2} = 0$ for all $n \geq 0$ if and only if $\alpha_{n+1}^2 = 2 - \frac{1}{\alpha_n^2}$ for all $n \geq 0$.
(iii) Note that W_α is 3-isometry if and only if $\gamma_n - 3\gamma_{n+1} + 3\gamma_{n+2} - \gamma_{n+3} = 0$ for all $n \geq 0$ if and only if $\alpha_{n+2}^2 = 3 - \frac{3}{\alpha_{n+1}^2} + \frac{1}{\alpha_n^2\alpha_{n+1}^2}$ for all $n \geq 0$. $\square$

THEOREM 2.4. *If* W_α *is m-isometry and the weight sequence* α *is convergent, then the limit of* α *must be 1.*

Proof. Suppose $\lim \alpha_n = a$ and $\epsilon > 0$ was given. First note that the equation (2.1) is equivalent to

$$1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \alpha_n^2 \cdots \alpha_{n+k-1}^2 = 0$$

for all $n \geq 0$. Since $\lim \alpha_n = a$, we can see that $|1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} a^{2k}| < \epsilon$ for sufficiently large n. But since $1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} a^{2k} = (1 - a^2)^m$, we have the desired result. $\square$

J. Stampfli [8] showed that for subnormal weighted shifts W_α, a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [7] that the shift with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \cdots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [6] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + s\,W_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [5, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If* W_α *is quadratically hyponormal and* $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ *for some* $n \geq 0$*, then* $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$*, i.e.,* W_α *is subnormal.* Furthermore, in [5, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality. Y. Choi [4] improved this result, that is, if W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then W_α is flat. Moreover, Y. Choi [4] also showed that if W_α is polynomially hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then W_α is flat.

Proposition 2.5. *(Propagation) Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$.*

(i) ([8, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*
(ii) ([5, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*
(iii) ([4, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*
(iv) ([4, Theorem 2]) *Let W_α be polynomially hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*

We now show that 2-isometric weighted shift operator is isometry if two equal weights are presented.

Theorem 2.6. *If W_α is 2-isometry with $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$ then $\alpha_n = 1$ for all $n \geq 0$ i.e., W_α is isometry.*

Proof. By Corollary 2.3 (ii), we can see that $\alpha_n = \alpha_{n+1} = 1$. Thus, by again Corollary 2.3 (ii), we have $\alpha_n = 1$ for all $n \geq 0$. □

Theorem 2.7. *If W_α is 2-isometry then the weight sequence α is decreasing and hence converges to 1.*

Proof. If W_α is 2-isometry, observe that $\alpha_{n+1}^2 = 2 - \frac{1}{\alpha_n^2} \leq \alpha_n^2$ for all $n \geq 0$. By the Monotone Convergence Theorem, the weight sequence α converges. By Theorem 2.4 the limit should be 1. □

Corollary 2.8. *For a weighted shift W_α, if $\alpha_n < 1$ for some $n \geq 0$ then W_α is not 2-isometry.*

Proof. Since the weight sequence α is decreasing, if $\alpha_n < 1$ for some $n \geq 0$, then the weight sequence α cannot converge to 1. Thus, W_α is not 2-isometry. □

Now, we can give an example which is 2-isometry but not hyponormal(and hence not subnormal).

Example 2.9. *Let W_α be the weighted shift with the weight sequence $\alpha \equiv \sqrt{\frac{n+2}{n+1}}$. Then W_α is 2-isometry but not hyponormal.*

Proof. Since $\gamma_n = n+1$ for all $n \geq 0$, we have $\sum_{k=0}^{2}(-1)^k\binom{2}{k}\gamma_{n+k} = \gamma_n - 2\gamma_{n+1} + \gamma_{n+2} = n+1-2(n+2)+n+3 = 0$. Thus, W_α is 2-isometry. However, W_α is not hyponormal because the weights are decreasing. □

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